J Glob Optim (2008) 40:65–70
DOI 10.1007/s10898-007-9182-9

Lagrange multipliers and infinite-dimensional equilibrium problems

Patrizia Daniele

Received: 29 May 2007 / Accepted: 29 May 2007 / Published online: 7 July 2007

Abstract We prove the existence of the Lagrange multipliers for a constrained optimization problem, being the constraint set given by the convex set which characterizes the most important equilibrium problems. In order to obtain our result, we'll make use of the new concept of quasi relative interior.

Keywords Lagrange multipliers · Separation theory · Equilibrium problems · Quasi relative interior

1 Introduction

For many network equilibrium problems such as the transportation, the migration, the vaccination, the spatial price markets (both in the price and in the quantity formulation), the electric power supply chain, the financial markets as well as the Internet networks, the convex set $\mathbb{K}$ of the constraints is of the following type (see [2,3]):

$$\mathbb{K} = \Bigg\{ x \in L^2([0,T],\mathbb{R}^q): \ x(t) \geq 0 \text{ a.e. in } [0,T], \sum_{i=1}^{q} \xi_{ji} x_i(t) = \rho_j(t), \\ j = 1,\dots,l \text{ a.e. in } [0,T], \quad \xi_{ji} \in \{0,1\}, \quad i \in \{1,\dots,q\} \Bigg\}$$

provided that each index i is such that there exists an index j^* for which $\xi_{j^*i} = 1$ and $\xi_{ji} = 0$ $\forall j \neq j^*$. If we define the matrix $\Phi = \{\xi_{ji}\}_{\substack{j=1,\dots,l \\ i=1,\dots,q}}$ and the vector $\rho(t) = [\rho_1(t),\dots,\rho_l(t)]^T$, then, for brevity, the constraints $\sum_{i=1}^{q} \xi_{ji} x_i(t) = \rho_j(t), \ \ j = 1,\dots,l,$ a.e. in $[0,T]$ can be rewritten as $\Phi x(t) = \rho(t)$ a.e. in $[0,T]$.

P. Daniele (✉)
Department of Mathematics and Computer Science, University of Catania, Viale A. Doria 6, 95125 Catania, Italy
e-mail: daniele@dmi.unict.it

The aim of this paper is to prove the following theorem.

Theorem 1 *Let us consider the optimization problem*

$$\min_{x\in\mathbb{K}} f(x) = f(x^0),$$

where $f : L^2([0,T],\mathbb{R}^q) \to \mathbb{R}$ *is a convex and Gâteaux differentiable functional. Then, there exist the Lagrange multipliers* $\overline{\lambda} \in L^2([0,T],\mathbb{R}^q)$, $\overline{\lambda} \geq 0$, $\overline{\mu} \in L^2([0,T],\mathbb{R}^l)$ *such that the point* $(x_0, \overline{\lambda}, \overline{\mu})$ *is a saddle point of the Lagrange functional*

$$\begin{aligned}\mathcal{L}(x,\lambda,\mu) &= f(x) - \sum_{i=1}^{q}\int_0^T \lambda_i(t)\,x_i(t)\,\mathrm{d}t - \sum_{j=1}^{l}\int_0^T \mu_j(t)\left(\sum_{i=1}^{q}\xi_{ji}x_i(t) - \rho_j(t)\right)\mathrm{d}t \\ &= f(x) - \int_0^T \langle\lambda(t), x(t)\rangle_q\,\mathrm{d}t - \int_0^T \left\langle \mu(t), \sum_{i=1}^{q}\xi_{ji}x_i(t) - \rho_j(t)\right\rangle_l \mathrm{d}t,\end{aligned}$$

$x \in L^2([0,T],\mathbb{R}^q),\ \lambda \in L^2([0,T],\mathbb{R}^q),\ \lambda \geq 0,\ \mu \in L^2([0,T],\mathbb{R}^l)$ *and*

$$\int_0^T \langle\overline{\lambda}(t), x^0(t)\rangle_q\,\mathrm{d}t = 0$$

that is $\overline{\lambda}_i(t),\ x_i^0(t) = 0\ \forall i = 1,\ldots,q$, *a.e. in* $[0,T]$.

Since the interior of the set $\mathbb{K}$ is empty, in order to show the above result, we use the new concept of *quasi relative interior* and the existence theorem shown in [4,5]. We also remark that in the paper [1] the authors consider a different constraint qualification assumption, namely the so-called closed convex cone qualification.

We recall the main result in [4], which we will apply to our case. Let X be a real normed space and S a nonempty subset of X; let $(Y, \|\cdot\|)$ be a real normed space partially ordered by a convex cone C; let $f : S \to \mathbb{R}$ and $g : S \to Y$ two functions such that the function

$$(f,g) : S \to \mathbb{R}\times Y$$

is convex-like with respect to the product cone $\mathbb{R}_+ \times C$ of $\mathbb{R}\times Y$. Let Z be a real normed space, $h : X \to Z$ an affine-linear mapping and assume that the constraint set is given by

$$\mathbb{K} = \{x \in S : g(x) \in -C,\ h(x) = \theta_Z\}.$$

Consider the constraint optimization problems:

$$\min_{x\in\mathbb{K}} f(x) \tag{1}$$

and

$$\max_{\substack{u\in C^*\\ v\in Z^*}} \inf_{x\in S}\,[f(x) + \langle u, g(x)\rangle + \langle v, h(x)\rangle], \tag{2}$$

where $C^* = \left\{u \in Y^* :\ \langle u, y\rangle \geq 0\ \forall y \in C\right\}$ is the dual cone of C and Z^* is the dual space of Z.

We introduce now an additional hypothesis (see [4]):

a point $x^0 \in \mathbb{R}$ satisfies *Assumption S* if it results:

$$T_{\widetilde{M}}(f(x^0), \theta_Y, \theta_Z) \cap \left\{\mathbb{R}^-, \theta_Y, \theta_Z\right\} = \emptyset,$$

where

$$\widetilde{M} = \{(f(x) + \alpha, g(x) + y, h(x)) : x \in S \setminus \mathbb{R},\ \alpha \geq 0,\ \ y \in C\}.$$

In [4] the authors proved the following result:

Theorem 2 *Let X be a real normed space and S be a linear subset of X; let $(Y, \|\cdot\|_Y)$ a partially ordered real normed space with convex ordering cone C and let $(Z, \|\cdot\|_Z)$ a real normed space. Let $f : S \to \mathbb{R}$ be a given functional and let $g : S \to Y$, $h : X \to Z$ be given mappings such that the function $(f, g) : S \to \mathbb{R} \times Y$ is convex–like with respect to the cone $\mathbb{R}_+ \times C$ of $\mathbb{R} \times Y$ and h is an affine–linear mapping. Let the set $\mathbb{R} = \{x \in S :\ g(x) \in -C,\ \ h(x) = \theta_Z\}$ be nonempty and let us assume that* qri $C \neq \emptyset$, cl $(C - C) = Y$, cl $h(S - S) = Z$ *and there exists $\hat{x} \in S$ with $g(\hat{x}) \in -$* qri C *and $h(\hat{x}) = \theta_Z$. If* Assumption S *is fulfilled at the extremal solution $x^0 \in \mathbb{R}$ to problem (1), then also problem (2) is solvable and, if $\bar{u} \in C^*$, $\bar{v} \in Z^*$ are the extremal points of problem (2), it results that*

$$\langle \bar{u}, g(x^0)\rangle = 0$$

and the extrema of the two problems are equal.

Using Theorem 2 we are able to prove Theorem 1 as shown in the following section.

2 Proof of Theorem 1

In our case, we have:

$$X = S = L^2([0, T], \mathbb{R}^q), \quad Y = L^2([0, T], \mathbb{R}^q),$$

$$C = \left\{x \in L^2([0, T], \mathbb{R}^q) :\ x(t) \geq 0 \text{ a.e. in } [0, T]\right\},$$

$$Z = L^2([0, T], \mathbb{R}^l), \quad g(x) = -x, \quad h(x) = [\Phi x - \rho].$$

Moreover, f is convex by assumption, g is convex by definition and h is an affine-linear mapping by definition; $\mathbb{K}$ is a nonempty set, qri $(C) = \left\{x \in L^2([0, T], \mathbb{R}^q) :\ x(t) > 0 \text{ a.e. in } [0, T]\right\}$, cl $(C - C) = L^2([0, T], \mathbb{R}^q)$ since $x(t) = x^+(t) - x^-(t)$ and, finally, $g(\hat{x}) \in -$ qri C. So, we only have to prove that also *Assumption S* holds true.

Since $f(x^0) = \min_{x \in \mathbb{K}} f(x)$ and is Gâteaux differentiable, it results:

$$f(x) - f(x^0) \geq \langle f'(x^0), x - x^0\rangle, \quad \forall x \in \mathbb{R}^q$$

and

$$\langle f'(x^0), x - x^0\rangle \geq 0, \quad \forall x \in \mathbb{K}. \tag{3}$$

We want to prove that variational inequality (3) yields that a.e. in $[0, T]$, $\forall j = 1, \ldots, l$, $\forall r, s$ such that $\xi_{jr} = \xi_{js} = 1$ it results:

$$f_f'(x^0(t)) > f_s'(x^0(t)) \implies x_r^0(t) = 0. \tag{4}$$

In fact, let us assume that condition (4) is not verified. Then, there exist a subset $E \subseteq [0, T]$: $m(E) = 0$, an index $j \in \{1, \ldots, l\}$ and a pair r, s of paths such that $\xi_{jr} = \xi_{js} = 1$ with $f_r'(x^0) > f_s'(x^0)$, but $x_r^0(t) > 0$ in E. Let us consider a vector $\tilde{x} \in \mathbb{K}$ such that:

$$\tilde{x}(t) = x^0(t), \quad \forall t \in [0, T]\backslash E \quad \text{and} \quad \tilde{x}_h(t) = \begin{cases} x_h^0(t) & \text{if} \quad h \neq r, s \\ 0 & \text{if} \quad h = r \\ x_r^0(t) + x_s^0(t) & \text{if} \quad h = s \end{cases} \qquad \forall t \in E.$$

Hence, variational inequality (3) becomes:

$$\begin{aligned} \langle f'(x^0), \tilde{x} - x^0 \rangle &= \int_E \sum_{h=1}^{q} f_h'(x^0)(\tilde{x}_h - x_h^0)\, \mathrm{d}t + \int_{[0,T]\backslash E} \sum_{h=1}^{q} f_h'(x^0)(\tilde{x}_h - x_h^0)\, \mathrm{d}t \\ &= \int_E \left[f_s'(x^0) - f_r'(x^0) \right] x_r^0(t)\, \mathrm{d}t < 0, \end{aligned}$$

which is an absurdity. Therefore, the equilibrium condition (4) holds true.

From (4) it derives, setting

$$\mu_j(t) = \min \left\{ f_h'(x^0(t)) : h = 1, \ldots, q \text{ such that } \xi_{hj} = 1 \right\} \in L^2([0, T])$$

for $j = 1, \ldots, l$ and a.e. in $[0, T]$ and for all $r = 1, \ldots, q$ such that $\xi_{rj} = 1$:

- if $x_r^0(t) > 0$, then $f_r'(x^0(t)) = \mu_j(t)$;
- if $x_r^0(t) = 0$, then $f_r'(x^0(t)) \geq \mu_j(t)$.

We are now able to show that *Assumption S* holds true. To this end, let

$$y = \lim_{n \to \infty} \lambda_n \left[f(x_n) + \alpha_n, -x_n + y_n, \Phi x_n(t) - \rho(t) \right] - \left[f(x^0), \theta_{L^2([0,T],\mathbb{R}^q)}, \theta_{L^2([0,T],\mathbb{R}^l)} \right]$$

with

$$\lambda_n > 0, \quad \lim_{n \to \infty} (f(x_n) + \alpha_n) = f(x^0), \quad \lim_{n \to \infty} (-x_n + y_n) = \theta_{L^2([0,T],\mathbb{R}^q)},$$

$$\lim_{n \to \infty} (\Phi x_n(t) - \rho) = \theta_{L^2([0,T],\mathbb{R}^l)}, \quad \alpha_n \geq 0, \quad y_n \in C, \quad x_n \in X.$$

First, let us prove that

$$\lim_{n \to \infty} \lambda_n \left[\Phi y_n(t) - \rho \right] = \theta_{L^2([0,T],\mathbb{R}^l)}.$$

In fact,

$$\begin{aligned} \lambda_n \left[\Phi y_n(t) - \rho \right] &= \lambda_n \left[\Phi y_n(t) - \Phi x_n(t) + \Phi x_n(t) - \rho \right] \\ &= \lambda_n \left[\Phi \left(y_n(t) - x_n(t) \right) \right] + \lambda_n \left(\Phi x_n(t) - \rho \right) \end{aligned}$$

and the last two terms converges to zero. Hence,

$$\lim_{n \to \infty} \lambda_n \left[\Phi y_n(t) - \rho \right] = \theta_{L^2([0,T],\mathbb{R}^l)}.$$

Now, we have to prove that

$$\lim_{n \to \infty} \lambda_n \left(f(x_n) + \alpha_n - f(x^0) \right) \geq 0.$$

Since $\alpha_n \geq 0$, it is sufficient to prove that

$$\lim_{n \to \infty} \lambda_n \left(f(x_n) - f(x^0) \right) \geq 0.$$

We have:

$$\begin{aligned}
&\lambda_n\left(f(x^n)-f(x^0)\right)\geq \lambda_n\langle f'(x^0), x^n-x^0\rangle\\
&\quad=\lambda_n\langle f'(x^0), x^n-y^n\rangle+\lambda_n\langle f'(x^0), y^n-x^0\rangle\\
&\quad=\langle f'(x^0), \lambda_n(x^n-y^n)\rangle+\lambda_n\int_0^T\sum_{j=1}^{l}\sum_{\xi_{rj}=1} f'(x^0(t))(y_r^n(t)-x_r^0(t))\,\mathrm{d}t\\
&\quad=\langle f'(x^0), \lambda_n(x^n-y^n)\rangle+\lambda_n\int_0^T\sum_{j=1}^{l}\left\{\sum_{\substack{\xi_{rj}=1\\ x_r^0(t)>0}} f_r'(x^0(t))(y_r^n(t)-x_r^0(t))\right.\\
&\qquad\left.+\sum_{\substack{\xi_{rj}=1\\ x_r^0(t)>0}} f_r'(x^0(t))\,y_r^n(t)\right\}\mathrm{d}t\geq\langle f'(x^0), \lambda_n(x^n-y^n)\rangle\\
&\qquad+\lambda_n\int_0^T\sum_{j=1}^{l}\left\{\sum_{\substack{\xi_{rj}=1\\ x_r^0(t)>0}}\mu_j(t)(y_r^n(t)-x_r^0(t))+\sum_{\substack{\xi_{rj}=1\\ x_r^0(t)>0}}\mu_j(t)\,y_r^n(t)\right\}\mathrm{d}t\\
&\quad=\langle f'(x^0), \lambda_n(x^n-y^n)\rangle+\lambda_n\int_0^T\sum_{j=1}^{l}\mu_j(t)\sum_{\xi_{rj}=1}\left(y_r^n(t)-\rho_j(t)\right)\mathrm{d}t\\
&\quad=\langle f'(x^0), \lambda_n(x^n-y^n)\rangle+\int_0^T\langle\mu(t), \lambda_n\left(\Phi y^n(t)-\rho(t)\right)\rangle_l\,\mathrm{d}t.
\end{aligned}$$

Now, we have $\lim\limits_{n\to\infty}\langle f'(x^0), \lambda_n(x^n-y^n)\rangle=0$, because $\lim_{n\to\infty}\lambda_n(-x^n+y^n)=\theta_{L^2([0,T],\mathbb{R}^q)}$ and $\lim\limits_{n\to\infty}\int_0^T\langle\mu(t), \lambda_n\left(\Phi y^n(t)-\rho(t)\right)\rangle_l\,\mathrm{d}t=0$, because $\lim\limits_{n\to\infty}\lambda_n\left(\Phi y^n(t)-\rho(t)\right)=\theta_{L^2([0,T],\mathbb{R}^l)}$.

Remark 1 If $x^0\in\mathbb{K}$ is a solution to the variational inequality

$$\text{“Find } x^0\in\mathbb{K}: \int_0^T\langle C(t,x^0(t)), x(t)-x^0(t)\rangle_q\,\mathrm{d}t\geq 0,\quad\forall x\in\mathbb{K}\text{"},$$

we can reduce such a variational inequality problem into problem (1), setting

$$f(x)=\int_0^T\langle C(t,x^0(t)), x(t)-x^0(t)\rangle_q\,\mathrm{d}t\quad\forall x\in\mathbb{K}$$

and observing that

$$f(x^0)=\min_{x\in\mathbb{K}} f(x)$$

(see, for details, [3,6]).

References

1. Bot, R.I., Wanka, G.: An alternative formulation for a new closed cone constraint qualification. Nonlin. Anal. **64**, 1367–1381 (2006)
2. Cojocaru, M.G., Daniele, P., Nagurney, A.: Projected dynamical systems and evolutionary variational inequalities via Hilbert spaces and applications. Jou. Optim. Th. Appl. **27**(3), 1–15 (2005)
3. Daniele, P.: Dynamic Networks and Evolutionary Variational Inequalities. Edward Elgar Publishing, UK (2006)
4. Daniele, P., Giuffré, S.: General infinite dimensional duality theory and applications to evolutionary network equilibrium problems. Optim. Lett. **1**(3), 227–243 (2007)
5. Daniele, P., Giuffré, S., Idone, G., Maugeri, A.: Infinite dimensional duality and applications. Mathematische Annalen **339**, 221–239 (2007)
6. Maugeri, A., Vitanza, C.: Time-dependent equilibrium problems. In: Chinchuluun, A., Migdalas, A., Pardalos, P.M., Pitsoulis, L. (eds.) Pareto Optimality, Game Theory and Equilibria. Nonconvex Optimization and its Applications Series (NOIA), pp. 505–524. Kluwer Academic Publishers, Dordrecht (2007)